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ESTIMATION THEORY WITH FRACTIONAL GAUSSIAN NOISE

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13. ABSTRACT (Maximum 200 words)

Physical processes often cannot be accurately modeled as purely deterministic mathematical processes because of the random aspects of their behavior. This unpredictable aspect is often statistically modeled as Gaussian white noise. This research developed an optimal estimator for parameters and states of systems driven by another type of noise, known as fractional Gaussian noise (FGN) processes. These stochastic processes can model systems containing long-term, slowly decreasing time-correlated random disturbances. Examples of processes that behave as fractional Gaussian noise processes are given. This report examines an estimator for an unknown parameter in a model represented by a form of FGN-driven stochastic differential equation. This parameter determines which equation out of a family of differential equations best fits the physical phenomena being modeled. A simulation of fractional Brownian motion (FBM) process and a state model driven by FGN was also developed for the purpose of testing the estimator. Their correspondence is discussed. The estimator was tested, using the simulations of FBM and the state model driven by FGN. The simulator has potential for applications for studying a physical process that cannot be replicated in a laboratory, although its behavior needs to be simulated for examination. An alternate estimator was also derived that was shown to possess the property of strong consistency.

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FOREWORD

Physical processes often cannot be accurately modeled as a purely deterministic mathematical process because of the existence of random aspects of their behavior. This unpredictable portion is often statistically modeled as Gaussian white noise. However, noise processes are not always characterized by the properties of Gaussian white noise. For example, a projectile whose motion is disturbed by air or water turbulence or a self-propelled projectile with unevenly mixed fuel may possess long-term slowly decreasing dependencies that may not be well represented by white noise. This research concerns an optimal estimator for parameters and states of systems driven by another type of noise, known as fractional Gaussian noise (FGN) processes. These stochastic processes can model systems containing long-term, slowly decreasing time-correlated random disturbances.

This report examines an estimator for an unknown parameter in a model represented by a form of FGN-driven stochastic differential equation. A simulation of fractional Brownian motion (FBM) process and a state model driven by FGN are also developed for the purpose of testing the estimator. FBM is a stochastic process that corresponds to FGN. This correspondence is discussed. The estimator is tested using the simulations of FBM and the state model driven by FGN. An alternate algorithm for estimating the unknown parameter is derived in this report. Also, estimating the single parameter that controls the time correlation of FBM is discussed.

Time and funding constraints did not permit further study to compare the methodologies reported here with more conventional methods for a specific application.

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CONTENTS

INTRODUCTION

The motion of a projectile or torpedo traveling through the air or water may be governed by a differential equation whose variable, commonly called its state, represents speed. Random disturbances affecting the flight or trajectory of the object may be caused by air or water turbulence, or, in the case of a self-propelled projectile, the disturbance could have originated from an uneven fuel mixture. Many physical processes, such as these, often cannot be accurately modeled as a purely deterministic mathematical process because of the existence of random aspects of their behavior. This unpredictable portion of the phenomena, often known as noise, is frequently statistically modeled as Gaussian white noise. However, noise processes are not always characterized by the properties of white noise. Thus, Gaussian white noise, which by definition assumes time independence may not form an accurate model. Furthermore, a constant, such as one representing friction divided by projectile mass may represent an unknown parameter in the differential equation (see Reference 1, p. 154). This investigation concerns optimal estimators for parameters and states of systems driven by a type of noise that possesses long-term, slowly decreasing time dependencies. This noise is known as fractional Gaussian noise (FGN), and it may represent systems with slowly decreasing dependencies, such as the example just described, more accurately than traditional white noise models.

The use of stochastic processes in the form of stochastic differential equations (differential equations disturbed by random noise) driven by white noise for modeling physical phenomena has been extensively studied for the case where the equations were driven by Gaussian white noise. Parametric estimators have been developed for parameters in some forms of stochastic differential equations driven by this type of noise. This report examines an estimator that was described in Reference 2 that can be used to estimate an unknown parameter of a model represented by a form of FGN-driven stochastic differential equation. A simulation of fractional Brownian motion (FBM) process and a state model driven by FGN was also developed for the purpose of testing the estimator. FBM is a stochastic process that corresponds to FGN. The estimator was tested, using the simulations of FBM and the state model driven by FGN. An alternate estimator was also derived in this report.

Another parameter that is very crucial in any models involving FBM is the single parameter known as H that characterizes FBM itself. The higher the value of H, the greater the time dependency of the FBM. In this research, H is assumed to be within the interval (0.5, 1.0) although it can be defined for all H within the interval (0.0, 1.0). The interpretation for processes where $H \geq 1$ is unclear (Reference 3, p. 943). After investigation, an algorithm was found in literature for estimating the single parameter needed to characterized FBM and its corresponding Gaussian noise process. This parameter determines the time correlation of the FBM and FGN.

BACKGROUND

This section provides some basic definitions and mathematical results needed for developing an optimal estimator for parameters and states of FGN-driven systems. These definitions and results can also help in understanding the reasons for potential applications of this work.

FBM DEFINITION

The most important background item is the definition of FBM, because this process is the originator of all processes in this research.

DEFINITION: Let B = {B(t): t ε R} be a standard Brownian motion (BM) process and R the set of real numbers. Fractional Brownian motion (FBM), symbolized by B_H, given H ε (0.5, 1) is defined as follows:

$$B_H(t) = \frac{1}{\Gamma(H+0.5)}\left(\int_{-\infty}^{0}(|t-s|^{H-0.5} - |s|^{H-0.5})dB(s) + \int_{0}^{t}|t-s|^{H-0.5}dB(s)\right) \tag{1}$$

where Γ is the gamma function, and s varies from negative infinity to t.

Hence, FBM is a stochastic integral of a deterministic function with respect to standard Brownian motion.

For H = 0.5, FBM is standard BM. Hence, FBM can be thought of as a generalization of standard BM. FBM is a zero mean normally distributed process with the following covariance function (R):

$$R_{B_H}(s,t) = \frac{V_H}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}) \tag{2}$$

$$V_H = \frac{-\Gamma(2-2H)\cos(pH)}{\pi H(2H-1)}; \tau = t-s \tag{3}$$

The distribution of B_H also has the property of self-similarity, which can formally be defined for FBM as follows:

$$\{B_H(at): t \in R\} \stackrel{d}{=} a^H \{B_{H(t)}: t \in R\};\ d = \text{distribution} \tag{4}$$

Equation (4) defines FBM self-similarity and states that the distribution of FBM looks the same regardless of the scale factor in time. The self-similarity property is also commonly known as the fractal property, and therefore the distribution of B_H can be said to possess the fractal property. One can see that by setting H = 0.5, BM also possesses this fractal property. This property is useful because some physical processes, or the random noise part of the physical processes, appear to behave as fractals.

An example of a physical system possessing random noise that behaves like FBM is random errors in communication channels as found in Reference 3. The errors in these channels may appear as groups of bursts, and this group of bursts is itself grouped in bursts, giving the appearance of the self-similarity property.

The FBM can also be defined by Equation (1) for H ε (0, 0.5). However, for modeling long-term dependencies, H is considered to be between 0.5 and 1.

FGN DEFINITION

FGN (symbolized by W_H) is considered to be the derivative of FBM, just as Gaussian white noise is considered to be the derivative of standard Brownian motion. In this sense, FGN corresponds to FBM and vice versa. Yet, analogous to the fact that Brownian motion is almost surely (a. s.) not differentiable, we have the following theorem:

THEOREM 1 (Reference 4): Fractional Brownian motion is a. s. not differentiable with respect to t.

The significance of this theorem is to show that in the strict mathematical sense, FGN (along with Gaussian white noise) does not exist as a true stochastic process. Yet, it is still desirable to use them as processes because of the useful properties they possess: FGN's long-term slowly decreasing time dependencies and 1/f properties (described in the next paragraph) and Gaussian white noise being time independent. These properties approximate physical systems. Furthermore, physical phenomena often behave as if they were derivatives of FBM or Brownian motion in spite of the fact that such derivatives do not truly exist. See Reference 3 for a description of how to define FGN in terms of an operator instead of a derivative of FBM.

FGN can be thought of as a stochastic process defined to be normally distributed with zero mean and the covariance function (R), which is defined in Equation (5), and spectral density function (S_{W_H}), defined in Equation (6):

$$R(t,s) = R_{W_H}(\tau) = V_H\, H\,(2H-1)|\tau|^{2H-2}$$

$$V_H = \frac{-\Gamma(2-2H)\cos(pH)}{\pi H(2H-1)}; \qquad \tau = t - s \tag{5}$$

$$S_{W_H}(w) = |w|^{1-2H}\,;\ w \in R,\ w \neq 0 \tag{6}$$

Equation (5) (the covariance function) shows that FGN exhibits the long-term dependency property of having slowly decreasing correlations. Because $-1.0 < 2H - 2 < 0.0$, the correlation is high for small t, which decreases slowly with increasing t since $2H - 2 > -1.0$. Moreover, since $-1.0 < 1 - 2H < 0.0$, the spectral density function (Equation (6)) shows that FGN has low-frequency, slowly decreasing spectral power for the same reason that FGN has high correlation for small t (decreasing slowly with increasing τ). Processes with such spectral densities are known as 1/f processes. Along with being normally distributed, these unique properties give FGN its potential for applications as mathematical/statistical models of physical processes.

Note that by setting τ to zero, Equation (5) shows that FBM has infinite variance. Also, notice that since $-1.0 < 1 - 2H < 0$,

$$\int_0^\infty S_{W_H}(w)\,dw = \int_0^\infty |w|^{1-2H}\,dw = \infty \tag{7}$$

Figure 1 contains graphs showing sample paths of standard Brownian motion and Gaussian white noise ($H = 0.5$). Figure 2 shows graphs of a sample path of FBM and FGN, where $H = 0.85$. These graphs are presented merely to show pictorially examples of the noise processes being studied. They are not part of the findings in this investigation.

SPECTRAL DENSITY

Since spectral density can be interpreted as the average spectral power per unit frequency (Reference 5, pp. 92-93), Equation (7) shows that FGN has infinite spectral power. Similarly, Gaussian white noise has infinite variance and infinite average spectral power. No natural process can possess such properties. Hence, in the physical sense as well as in the mathematical sense, neither FGN nor Gaussian white noise can exist. However, both of these pseudo-processes can form approximations of many real-life phenomena where the spectral power density is non-zero for a wide range of frequencies.

Gaussian white noise is well known in the literature for having a constant spectral density function. Therefore, it is often useful for representing processes with constant spectral power over a wide range. FGN is useful for representing phenomena in an approximate manner with low frequency power that is high near zero and slowly decreasing over a wide range before becoming zero. Note that processes with the long-term slowly decreasing time dependencies, as given by Equation (5) (covariance) used for "defining" FGN, can be shown to have the low frequency spectral power as also given in Equation (6).

One example of a physical process that behaves as a FGN is discharge from a river. These discharges may tend to exhibit clusters of high and low periods and therefore exhibit long-term dependencies. As a second example, the influence of sea wave action on the accuracy of projectiles fired from weapons on ships may also possess the long-term time dependency property of FGN. Further investigation needs to be conducted regarding this second example to study how closely it can be approximated by FGN. Taking the "derivative" of a model of the errors portion of the communications channel (as described previously) gives a third example of FGN. Also, the effects of air or water turbulence on projectiles or the effects of unevenly mixed fuel could be candidates for FGN representation. This was described in the introduction.

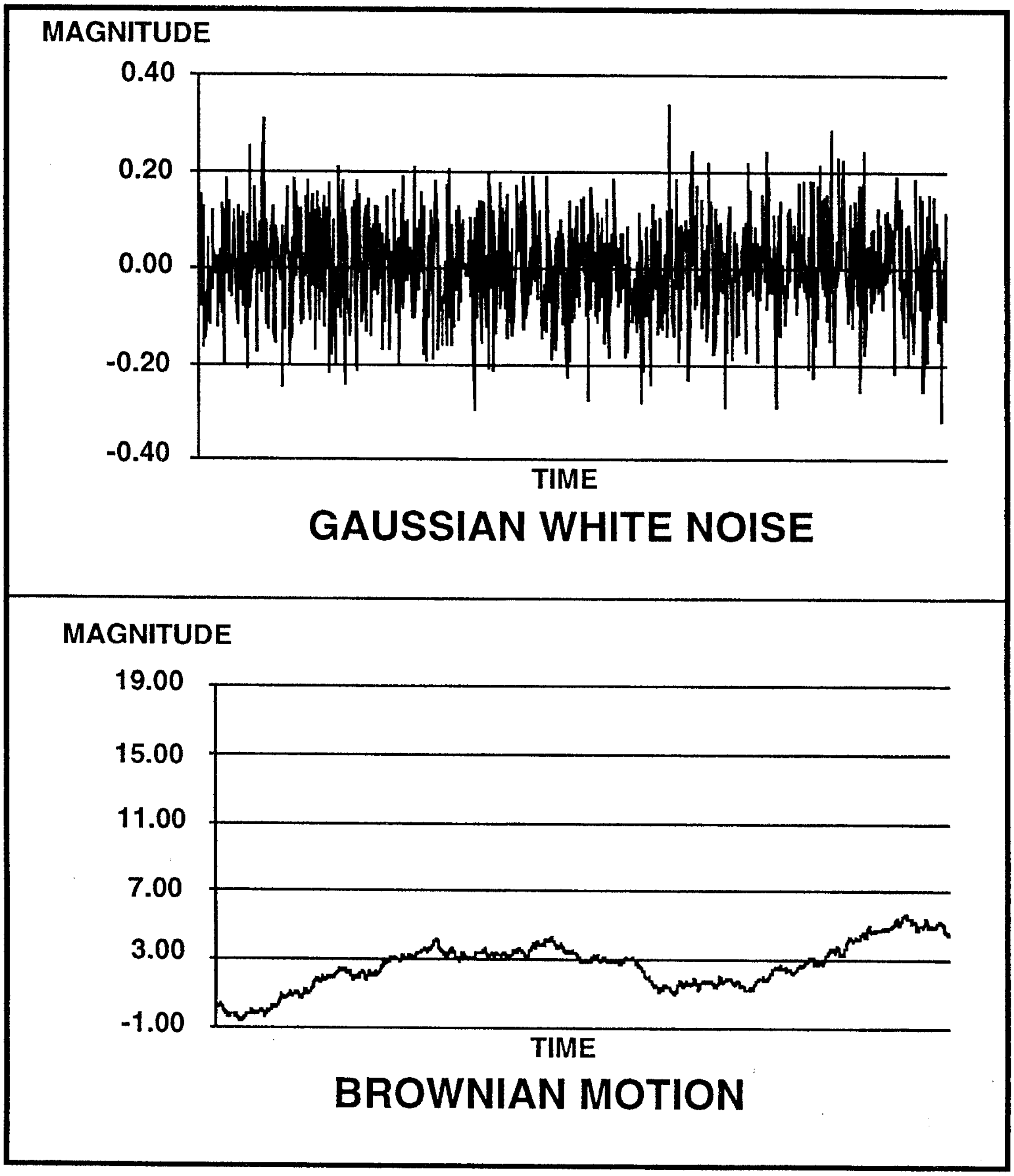

FIGURE 1. RANDOM NOISE I (H = 0.50)

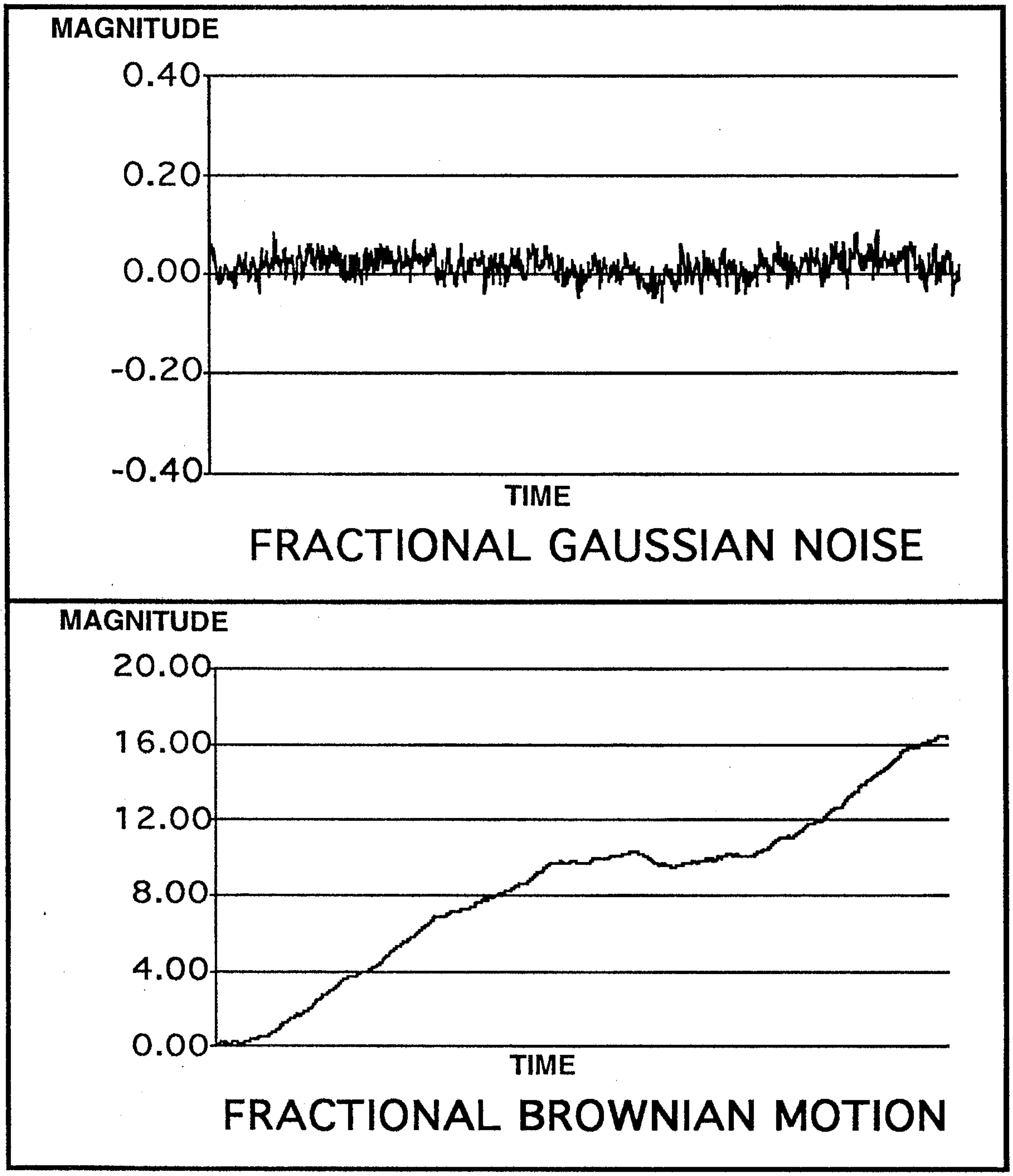

FIGURE 2. RANDOM NOISE II (H = 0.85)

Although FBM is not differentiable, a "pseudo-derivative" can be approximated by determining the ratio of an increment of B_H to an increment of time and thinking of it as a derivative in analyzing mathematical properties of the model.

MARTINGALE

Another important concept is the definition of a martingale. A martingale is a stochastic process where the expectation of its future state is precisely its present state. The precise definition follows:

DEFINITION: Given a pair (M(t), a(t): t ε T) where M(t) is a stochastic process that is measurable with respect to a(t), an increasing family of σ-algebras, then the pair is called a martingale if

$$E|M(t)| < \infty \qquad \text{and} \qquad E[M(t)|a(s)] = M(s)\,\forall t > s \tag{8}$$

The most common example of such a process is standard Brownian motion itself. See Reference 2 p. 6 for a definition of a σ-algebra.

A martingale has two useful properties: Future expectations can be described, and a martingale lends itself to well-defined stochastic integrals. If a process is not at least a generalization of martingales known as a local martingale, possible problems arise in defining integrals of another stochastic process with respect to this original non-martingale process. See Reference 5, p. 234, for a precise definition of a local martingale.

PROBLEM STATEMENTS AND APPROACH

OPTIMAL ESTIMATOR FOR PARAMETER θ

Developing the Estimator

The first problem was to develop an optimal estimator for the parameter θ in the following FGN-driven stochastic differential equation.

$$dX(t) = \theta X(t)dt + W_H(t)dt \tag{9}$$

With FGN not being a technically true stochastic process, perhaps a more accurate notation in the strict mathematical sense for the same equation is shown in Equation (10):

$$dX(t) = \theta X(t)dt + dB_H(t) \tag{10}$$

Because FBM is not differentiable, one may ask what is the meaning of this stochastic differential equation. The answer is that Equation (10) is a symbolic form of the following integral equation:

$$X(t) = \theta \int_0^t X(s)\,ds + B_H(t) \tag{11}$$

Comparing the stochastic differential equation with the random noise in the form of FGN with the equivalent stochastic integral equation with its random noise in the form of FBM further reinforces the concept that FBM and FGN correspond to one another.

The stochastic differential equation, Equation (10), with an unknown parameter θ is potentially useful in modeling physical systems whose behavior resembles that of a family of differential equations indexed by the parameter θ and whose random noise behaves like FGN. Hence, finding an estimate of the unknown parameter θ is one method of selecting which of the differential equations in such a family of equations best approximates the system being modeled.

Fitting Equation (10), or equivalently Equation (9), into the projectile example given in the introduction, θ represents the frictional constant divided by the projectile mass, X represents the projectile velocity, and W_H represents the random disturbance.

Methods have been established to derive an optimal estimator if the noise process is a square-integrable martingale. However, one can show the following theorem:

THEOREM 2 (Reference 2, pp. 47-49): B_H is not a local martingale for $H \varepsilon (0.5, 1)$.

Therefore, B_H cannot be a martingale for $H \varepsilon (0.5, 1)$ because local martingales are more general than martingales. A direct proof that B_H is not a martingale for $H \varepsilon (0.5, 1)$ was also presented in Reference 2. Hence, a method of estimating the parameter in Equation (10) needed to be developed. As already noted, when $H = 0.5$, B_H is standard Brownian motion, which is a martingale.

One of the methods developed in Reference 2 for estimating θ was based upon a pseudo-least-squares estimator shown in Christopeit (Reference 6). In Christopeit's "quasi-least-squares" method, the random noise is still assumed to be a martingale. Otherwise, however, one can show that the model represented by Equation (10) can fit into the model given in Reference 6. Note that the quasi-least-squares method was derived as a continuous generalization of the conventional (discrete) least-squares estimator. A conventional least-squares estimator was derived without regard to random noise distribution. Also, the only places where the martingale property was used in developing the quasi-least-squares method were in defining stochastic integrals and in determining asymptotic properties such as consistency. Hence, the least-squares estimator itself will be mathematically correct as long as the terms within the estimator, involving stochastic integrals, are well-defined.

The modified Christopeit's "quasi-least-squares" solution is shown in Equation (12):

$$\hat{\theta}(t) = \frac{\int_0^t X(s)dX(s)}{\int_0^t X^2(s)ds} \tag{12}$$

See Reference 2, pp. 63-65, for a detailed explanation of this formula.

The denominator on the right-hand side of Equation (12) is merely a Riemann or Lebesgue integral. However, the interpretation of the integral in the numerator is not as clear. The form of the model's stochastic differential equation shows that the dX (s), located under the integral sign in the numerator, is actually equivalent to $\theta X(s)\, ds + dB_H(s)$; that is,

$$dX(s) = \theta X(s)ds + dB_H(s) \tag{13}$$

Hence, the integral in the numerator of the right-hand side implicitly contains two integrals: an integral of $\theta X(s)$ with respect to s, which can be interpreted as a Riemann or Lebesgue integral, and a stochastic integral with respect to B_H, which is in the following form

$$\int_0^t X(s)dB_H(s) \tag{14}$$

These facts were explained in Reference 2, pp. 64-65. Since B_H is neither a square-integrable martingale or a local martingale, investigation was required to determine whether this integral can be shown to be well-defined. An argument developed in Reference 2, pp. 53-62, shows that this integral is indeed well defined. This finding was essential to assure the estimator's numerator is meaningful.

The formula for the modified Christopeit's quasi-least-squares estimation method was coded in FORTRAN for running on a VAX computer system.

Testing the Estimator

To test the program, a method for simulating the X given parameter θ must be found. The algorithm that was derived in this research was in the form of a Monte Carlo simulation of the solution to the model shown in Equation (10). The solution to this stochastic differential equation follows for X (0) = 0:

$$X(t) = \int_0^t e^{(t-s)\theta} dB_H(s) \tag{15}$$

The derivation of Equation (15) is explained in Reference 2, pp. 37-40. Note that because the integrand of this integral is a deterministic function, there is no ambiguity concerning whether the integral is well-defined.

Because X is an integral with respect to FBM, in order to simulate X in a Monte Carlo manner, a method of simulating the FBM for any H such that H ε (0.5, 1) must also be found. A Monte Carlo simulation of B_H was developed as a multivariate normally distributed random vector of dimension 1000 or 2000 where each component B_H (i) (i = 1,..., n; n = 1000 or 2000) represented a different time, i Δt, of B_H. Δt was a constant to be read in as an input representing the desired increment in time between successive components of the random vector.

Ideally, one desires to have the highest possible dimension in the random vector because the higher the dimension, the greater the amount of data, which improves the ability to test for the convergence of the estimator. However, because of capacity limitations of the VAX computer, either 1000 or 2000 data points are near the maximum amount of data for inputting into the computer for this particular project. A dimension of 2000 was used for analysis in single precision, but, because double precision requires more memory, a dimension of only 1000 was used for double precision analysis. The mean of the normally distributed random vector was set to be zero because FBM by definition has zero mean. The covariance matrix was designed so that each entry represented a covariance of the FBM between two of the times.

Given the simulated FBM and noting that X's solution was an integral with respect to FBM, X was simulated as a numerical approximation of the integral with respect to the simulated FBM. A numerical approximation of a Monte Carlo simulation of X thus was obtained.

Finally, noting that Christopeit used the martingale property for determining consistency and knowing that B_H lacks this property, a third task was to find an alternate method for determining consistency. After extensive study, an analytic algorithm has not yet been found. Therefore, one possible method may be to empirically test the consistency using the simulated X. The empirical testing of consistency has not yet been performed. Performing an empirical test of consistency is currently difficult because the VAX's memory capability limits the dimension of the multivariate random vector FBM simulation to 1000 or 2000 (as previously explained).

An alternate estimator is also derived in this report.

FBM PARAMETER H ESTIMATION

Another problem for investigation was to estimate the FBM parameter H, assuming that H ε (0.5, 1). In the Equation (10) model, the FBM, B_H (t), was assumed to be unknown for nonnegative t because only X (t) or increments of X (t) were assumed to be observable. However, in some applications, B_H (t), may possibly lend itself to observation for negative t. Hence, two data sources may be used to estimate H.

The most accurate way is to estimate H directly from the observed data for B_H (t) for negative t. If this data is unavailable, the alternate method is to first estimate θ and then solve for the B_H (t) in the integral form of the stochastic differential equation model:

$$B_H(t) = X(t) - \hat{\theta} \int_0^t X(s)\,ds \tag{16}$$

The estimated B_H (t) obtained in this manner can then be used to estimate H, although the accuracy of the estimated H from data obtained by this alternate method is unknown and needs to be studied, probably by empirical methods.

One method for estimating H (given the FBM data) is described in Reference 7. This method is based upon a least-squares fit to the incremental B_H data given by

$$V(j,k) = \sum_{i=1}^{k} [\Delta B_H((j+i)\Delta t) - m(\Delta B_H((j+i)\Delta t))]^2 \tag{17}$$

where j is any positive integer. The Δ represents the increment of B_H between $(j+i)\Delta t$ and $(j+i-1)\Delta t$. H is estimated as being one-half (1/2) the slope of a straight line fitted using the least-squares method to data given in the form of log V (j, k) versus log (k) as shown in Equation (18).

$$\log V(j,k) = 0.5H \log k + C \tag{18}$$

C is a constant.

KALMAN FILTER GENERALIZATION FOR X ESTIMATOR

Another research problem also dealt with the same FGN-driven stochastic differential equation, Equation (9); that is, the same equation with FGN as the random noise. One assumed in this problem that parameter θ was known but the state X (t) was unknown. Moreover, an observation model with the random component being the traditional Gaussian white noise was assumed to be available. The problem was to investigate the possibility of generalizing the Kalman filter into an estimator that can estimate the state X in spite of the fact that X was driven by FGN instead of Gaussian white noise.

ACHIEVEMENTS

The previous section defined problems that must be considered for FGN Estimation Theory. Approaches for resolving those problems were also examined. This section details the procedures and achievements.

OPTIMAL ESTIMATOR FOR PARAMETER θ

As stated in the previous section, an estimator for θ in the stochastic differential equation, Equation (10) was developed in the form of the modified Christopeit's generalized least-squares fit (Equation (12)). The coding for this estimator and every computer program in this research project was written in FORTRAN using the VAX computer system. This estimator was checked by using the simulation of X that was a function of the Monte Carlo simulation of B_H. Reference 8 described the software package that was used to generate a normally distributed

random vector. Theorem 2 showed that B_H is not a local martingale for H ε (0.5, 1). Also, stochastic integrals involved in the estimator can be shown to be well-defined (see previous section or Reference 2, pp. 53-62, pp. 64-65). This is necessary to ensure that the estimator itself is well-defined.

Denoting the single parameter of FBM again by H and assuming H is in the interval [0.5,1), this algorithm worked well for H = 0.5 for three conditions:

(1) The unknown parameter is positive and less than or equal to approximately 4.6.
(2) The increment of time is set to 0.001 units.
(3) The array is allowed to hold a sequence of 2000 different times (see Table 1).

The numbers as shown for θ in Table 1 were chosen merely as representations of possible parameters and are not of any particular significance. If the parameter was set to a value much larger than approximately 4.6, the program aborted because X's magnitude was too large.

TABLE 1. ESTIMATION OF θ

VALUE OF H	θ = 1.0	θ = 4.5	θ = -2.0	θ >> 4.6
0.5	ε = 0.09	ε = 0.01	ε = 0.08	program aborts
0.75	ε = 0.005	ε = 0.1	ε = 1.9	program aborts

It was noted that FBM becomes standard Brownian motion when H = 0.5. A true FBM that is not merely Brownian motion was simulated with H = 0.75. In this case, double precision was needed for the program to work. However, double precision necessitated greater computer memory than single precision. Therefore, the array had to be decreased back to 1000 elements. Also, the increment of time was changed to 0.01 units. For positive valued parameters not exceeding 4.6, reasonably accurate estimates resulted. However, the results were not as favorable for the negative valued parameter, probably because for the negative value parameters, 1000 different time elements did not comprise a sufficiently large sampling of the model. However, in double precision, the VAX's space limitation did not allow an array much larger than 1000. The FORTRAN listing of the double precision versions of the simulator and estimator programs is shown in Appendix A. Table 1 also gives results for H = 0.75. Further investigation of the handling of this array problem is needed to determine alternative possibilities for testing the estimator.

Another method for simulating the FBM for use in obtaining X is to first perform a Monte Carlo simulation of independent Gaussian random numbers, which can be used to represent dB (s) for time s. Then the FBM can be simulated using either the given formula defining B_H (see Equation (1)) or another equation for FBM restricted to time within a closed and bounded set [0, T], such that $T < \infty$; that equation is shown in the next section. Then the X can be simulated in the same manner already described.

Refer to Equation (12) for the estimator. Suppose that the experimental data is "close" to being continuous and analog, enabling readings for extremely small increments of time. As shown, the estimator involves a ratio of two stochastic integrals, defined to be the quadratic mean (q. m.) limits of summations over partitions of [0, t]. This convergence is analyzed below.

$E[X^2(s)]$ is a finite and continuous function for all $s\varepsilon[0,t]$ for finite t with the set [0, t] being Lebesgue measurable. Therefore, the integral in the denominator can be treated as a Lebesgue integral, according to Reference 5, p. 45, instead of as a quadratic mean integral. Furthermore, Equation (15) showed X as an integral of a continuous function with respect to a FBM, which is the sum of 2 integrals of continuous functions with respect to a Brownian motion. Brownian motion is equivalent to a process that is a. s. sample path continuous (Reference 9, pp. 66-67). This implies that X^2 must be equivalent to an a. s. continuous process. Thus, the integral can further be assumed to be a Riemann integral. The denominator as a Riemann integral is analyzed as follows:

Consider the partition of [0, t], $\{t_0 = 0, t_1,\ldots,t_n = t\}$. Let $I_1(t;\theta,n)$ represent the numerical summation approximating the integral of the denominator. Then $|I_1(t;q,n) - I_1(t;q,m)|\to 0$ as $n,m\to\infty$ and $\max(\Delta t)\to 0$ according to Cauchy's sequence theorem. This numerical integration can be repeatedly performed to obtain both $I_1(t;\theta,m)$ and $I_1(t;\theta,n)$, refining the partition with each iteration, assuming that the continuous experimental data allows such refinement, until $|I_1(t;\theta,n) - I_1(t;\theta,m)|$ no longer decreases.

Since the numerator is not equivalent to any Lebesgue or Riemann integral, analyzing the partition of [0, t] is more involved than analyzing the denominator. Also, an estimated value for H and a finite range of possible values of θ must be known. For this case, let $I_2(t,\theta,n)$ be the function approximating the integral in the denominator. Since $I_2(t,\theta,n)$ converges to the actual integral in q. m. instead of in the ordinary limit (as in the Lebesgue or Riemann integral), $|I_2(t;\theta,n)- I_2(t;\theta,m)|\to 0$ cannot be claimed. The q. m. convergence implies that $\sqrt{E[\,|I_2(t;\theta,n)- I_2(t;\theta,m)|^2]}\to 0$. However, without knowing the precise value for θ and given only a single sample path of data, $E[\,|I_2(t;\theta,n)- I_2(t;\theta,m)|^2]$ cannot be determined. Thus, the values for $E[\,|I_2(t;\theta,n)- I_2(t;\theta,m)|^2]$ must be determined for a known range of possible values of θ or by substituting the already estimated value. One can refine the partition $\{t_0=0, t_1,\ldots, t_n=t\}$ until $\sqrt{E[\,|I_2(t;\theta,n)- I_2(t;\theta,m)|^2]}$ no longer decreases for each θ and from this set choose the one with the most refined partition. Note that given a trial θ, $\sqrt{E[\,|I_2(t;\theta,n)- I_2(t;\theta,m)|^2]}$ can be obtained by two methods. One way is by mathematical derivation using Equation (15) for X(t) and the properties of expectations. The other is by Monte Carlo simulation of a set of sample paths of $I_2(t;\theta,n)- I_2(t;\theta,m)$ and finding the sample second moment.

The purpose of this description of convergence is only to describe its nature. In actuality, the practical way to check the partition for a given t is to merely select a partition and repetitively refine it until the estimated θ stabilizes.

An alternate estimator and its derivation are discussed below. It has the advantage of a. s. convergence, which will be justified. However, its disadvantage is that the value of the H parameter for the FBM is assumed to be already known, while the estimator just described does not make this assumption.

FBM PARAMETER H ESTIMATING THE STATE X

An algorithm from Reference 7 that is an estimator for the value of the H parameter, given FBM data, was described in the previous section. Future plans to implement this estimator in FORTRAN and to test it using the simulated FBM have been proposed. Also proposed are plans to test this estimator for its performance when the data are available only in the form of the simulated X, assuming that the value of the parameter θ is known.

KALMAN FILTER GENERALIZATION FOR ESTIMATING THE STATE X

Finally, considerable research has been performed in an attempt to derive a generalization of the Kalman filter. That filter can perform an optimal estimate of the state X for the model where the parameter θ was then assumed to be known (see Equation (10)). Recall that this problem is a generalization of the Kalman filter since the noise is FGN and not Gaussian white noise, which is the assumed noise of a state model to be estimated by the Kalman filter. At the present time, however, such a generalized filter has not yet been worked out. Whether such a generalized filter can be mathematically derived currently remains unknown.

ALTERNATE ESTIMATOR FOR PARAMETER θ

Suppose that the value of H is unknown and the model is as follows:

$$dX(t) = \theta X(t)dt + dB_H(t) \text{ for } t>0 \text{ and} \tag{19}$$

$$dX(t)=dB_H(t) \text{ otherwise}$$

The H parameter can then be estimated by using the data for negative time. With this estimated H, another algorithm, derived in Reference 2, can be used to estimate the θ. This algorithm was shown to be strongly consistent. In other words, the alternate estimator converges a. s. to the actual θ.

Now suppose that data are only available for $t \varepsilon [0,T]$ for finite T and that the value of H is assumed to be known. The following alternate algorithm that converges a. s. as finite T becomes infinitely large can be derived for the model:

$$dX(t) = \theta X(t)dt + dB_H(t) \quad t \varepsilon [0,T] \tag{20}$$

The following theorem must be used to derive this new estimator:

THEOREM 3 (Reference 3, pp. 951-952): Given the closed and bounded indexing set [0,t], there exists a standard Brownian motion process B_T for $t \varepsilon [0,T]$ such that for the FBM restricted to $t \varepsilon [0,T]$, symbolized by $B_{H|T}$, the following transformation holds:

$$B_T(t) = \frac{1}{\Gamma(1.5-H)} \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} dB_{H|T}(u)] \tag{21}$$

With the X in the model, let

$$Y(t) = \frac{1}{\Gamma(1.5-H)} \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} dX(u) \tag{22}$$

This implies that:

$$Y(t) = \frac{1}{\Gamma(1.5-H)} [\theta \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} X(u) du + \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} dB_{H|T}(u)] \tag{23}$$

Theorem 3 implies that the second term for the expression of Y must be Brownian motion. Hence,

$$Y(t) = \frac{1}{\Gamma(1.5-H)} [\theta \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} X(u)\, du + B_T(t)] \tag{24}$$

or

$$Y(t) = \theta R(t) + B_T(t) \tag{25}$$

where

$$R(t) = \frac{1}{\Gamma(1.5-H)} \int_0^t s^{H-0.5} d_s \int_0^s (s-u)^{0.5-H} u^{0.5-H} X(u) du \tag{26}$$

The formula for integration by parts is known to also hold true for integrals of a stochastic process with respect to time or integrals of a deterministic function with respect to a stochastic process (see Reference 9, p. 88). This fact is needed to find the derivative of R; R is needed for using the estimator that is now being derived. Another fact used to find dR/dt is Theorem 4:

THEOREM 4: Let f and g be two deterministic and differentiable functions in [0,T]. Let X (not necessarily the same X of the model in this report) be any stochastic process with a covariance function that is of bounded variation in the set [0,T] x [0,T]. Then

$$\int_0^t f(s) d[g(s)X(s)] = \int_0^t f(s)X(s)\dot{g}(s)ds + \int_0^t f(s)g(s)dX(s) \tag{27}$$

Proof: By the stochastic version of the integration by parts formula,

$$\int_0^t f(s)d[g(s)X(s)] = [f(s)g(s)X(s)]_0^t - \int_0^t g(s)X(s)\dot{f}(s)\,ds \tag{28}$$

Moreover, by using the stochastic version of integration by parts again,

$$\int_0^t f(s)g(s)dX(s) = [f(s)g(s)X(s)]_0^t - \int_0^t g(s)X(s)\dot{f}(s)ds - \int_0^t f(s)X(s)\dot{g}(s)ds \tag{29}$$

Therefore,

$$\int_0^t f(s)X(s)\dot{g}(s)ds + \int_0^t f(s)g(s)dX(s) = [f(s)g(s)X(s)]_0^t - \int_0^t g(s)X(s)\dot{f}(s)ds \tag{30}$$

Note that both the left and right sides of the equation in the theorem statement are equal to the same expression, meaning that the equation must be true. Therefore, the proof of Theorem 4 is now complete.

Also, if a process, say Q, is either sample path or q.m. differentiable,

$$\int_a^t f(s)dQ(s) = \int_a^t f(s)\left(\frac{dQ(s)}{ds}\right)ds \tag{31}$$

where the derivative is either q.m. or sample path, and f is an ordinary deterministic function. This fact can be proven by using the two variations of the stochastic integral by parts formula to show that either of the above two integrals are equal to the same thing.

With these concepts, Equation (26) for R(t) can now be manipulated to give a form that lends itself to deriving its derivative. Using Leibniz formula for derivatives on the differential term,

$$d_s \int_0^s (s-r)^{0.5-H} X(r)\,dr = \tag{32}$$

$$\lim_{\varepsilon \to 0} \left[\int_0^{s-\varepsilon} (0.5-H)(s-r)^{-0.5-H} r^{0.5-H} X(r)dr + \varepsilon^{0.5-H}(s-\varepsilon)^{0.5-H} X(s-\varepsilon)\right] ds$$

Note that the above integral is a function of s that is of bounded variation and thus has a derivative almost everywhere (a. e.) in s. Hence, the traditional sample path and the quadratic

mean derivative are a. e. the same in s. In analyzing the above differential term, one can think of it as a sample path differential that obeys the conventional laws of differential calculus a. e. With the conventional laws of calculus and uniform convergence in mind while considering d_s, the limit of d/ds and d/ds of the limit can be shown to be equal for the case being considered here. Explanations can be found in books on advanced calculus or real analysis such as Reference 10. Making the substitutions for integration by parts on the term with the integral in the above expression,

$$u = X(r)r^{0.5-H} \Rightarrow du = (0.5 - H)X(r)r^{-0.5-H}dr + r^{0.5-H}dX(r) \tag{33}$$

$$dv = (s-r)^{-0.5-H}dr \Rightarrow v = \frac{-1}{0.5-H}(s-r)^{0.5-H} \tag{34}$$

Equation (33) for du is merely a set of symbols from formal manipulations. It becomes meaningful when combined with the expression for v and integrated (using Theorem 4) as part of the integration by parts formula. Using these terms for u, v, du, and dv for integrating the term in the right-hand side of Equation (32) with the integral,

$$\lim_{\varepsilon\to 0}\Big[\int_0^{s-\varepsilon}(0.5-H)(s-r)^{-0.5-H}r^{0.5-H}X(r)dr + \varepsilon^{0.5-H}(s-\varepsilon)^{0.5-H}X(s-\varepsilon)\Big]ds =$$

$$\lim_{\varepsilon\to 0}\Big[-\varepsilon^{0.5-H}(s-\varepsilon)^{0.5-H}X(s-\varepsilon) + (0.5-H)\int_0^{s-\varepsilon}(s-r)^{0.5-H}X(r)r^{-0.5-H}dr +$$

$$\int_0^{s-\varepsilon}(s-r)^{-0.5-H}r^{0.5-H}dX(r) + \varepsilon^{0.5-H}(s-\varepsilon)^{0.5-H}X(s-\varepsilon)\Big]ds =$$

$$\Big[(0.5-H)\int_0^{s}(s-r)^{0.5-H}X(r)r^{-0.5-H}dr + \int_0^{s}(s-r)^{-0.5-H}r^{0.5-H}dX(r)\Big]ds \tag{35}$$

$$\therefore R(t) = \frac{1}{\Gamma(1.5-H)}\int_0^t s^{H-0.5}\Big[(0.5-H)\int_0^s(s-r)^{0.5-H}X(r)r^{-0.5-H}dr +$$

$$\int_0^s(s-r)^{-0.5-H}r^{0.5-H}dX(r)\Big]ds \tag{36}$$

Hence, taking the derivative of R(t) with respect to t gives the following:

$$\frac{dR(t)}{dt} = \frac{1}{\Gamma(1.5 - H)} t^{H-0.5}[(0.5 - H)\int_0^t (t-r)^{0.5-H} X(r) r^{-0.5-H} dr + \int_0^t (t-r)^{-0.5-H} r^{0.5-H} dX(r)] \tag{37}$$

Following the same arguments in Reference 2, pp. 71-76, for the derivation for the estimator in Equation (19), the estimator for the present model with t ε [0,T] can be shown to be as follows:

$$\hat{\theta}(T) = \frac{\int_0^T (dR(s)/ds)\, dY(s)}{\int_0^T [dR(s)/ds]^2 ds} \tag{38}$$

Values for Y can be derived from Equation (22). Furthermore, the derivative of R as shown in Equation (37) is in terms of X. Hence, everything in the estimator of θ can be obtained from X, which is the original given data. This means that estimator can be implemented to give a numerical estimation of θ.

Now consider an infinite class of closed and bounded sets $\{[0,T_n]: T_n$ finite, $n = 0,1,2,...,$ and $T_{n+1}>T_n\}$. Note that Theorem 3 implies that a Brownian motion exists for each $[0,T_n]$ such that the desirable properties are fulfilled. However, examine the formula in Theorem 3, which tells about the existence of Brownian motion given FBM in a finite interval. This formula is valid for both $[0,T_n]$ and $[0,T_{n+k}]$ for any positive integer k, and it shows that the sample path up to T_{n+k} of the interval when restricted to T_n is the same as the sample path of Brownian motion derived for the closed and bounded set $[0,T_n]$. Therefore, the estimator derived in Equation (38) is equivalent to an estimator for all finite T ε [0, ∞). Thus, considering whether asymptotic properties such as consistency exist is still meaningful in spite of the fact that the algorithm was derived for a finite index set. This algorithm can be shown to be strongly consistent; that is, it converges a. s. to the actual θ by the same argument given in Reference 2, pp. 77-86, for showing the strong consistency of the parametric estimator for the model given by Equation (19).

SUMMARY

The purpose of this research project was to find methods for modeling physical phenomena that contain an unknown constant parameter and random noises, characterized by long-term slowly decreasing time dependencies. The model being considered is in the form of a stochastic differential equation written as Equations (9) or (10) in this report. The unknown parameter is θ. With this purpose in mind, an estimator that estimates the unknown parameter was tested. The noise process being used in this project is fractional Brownian motion (FBM or B_H) or fractional Gaussian noise (FGN or W_H). The question concerning whether this estimator is convergent (converges to the actual value of the parameter) remains inconclusive from the tests. FBM and FGN are themselves characterized by a single parameter known as H. An algorithm that can be used to estimate this single parameter of FBM was also described in this report.

A Monte Carlo simulation of the FBM as a multivariate normally distributed random vector (where each component of the vector represents FBM at a different time) has been developed. This was in turn used to create a simulation of the variable called X, in the stochastic differential equation model, Equation (10). The simulation of X was necessary to test the results of the estimator for the parameter θ. Furthermore, the simulator of X itself has potentials for applications for studying a physical process that cannot be replicated in a laboratory, although its behavior needs to be studied.

In addition to the above, an alternate estimator was also derived to estimate the parameter θ. This estimator must assume prior knowledge of the value of the FBM parameter H but can be shown to converge to the true value of the parameter.

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APPENDIX A

FORTRAN LISTING OF THE SIMULATOR AND ESTIMATOR PROGRAMS

```
      PROGRAM ESTIMATION
C THIS PROGRAM IS IN DOUBLE PRECISION AND IS CALLED ESTIMATION2DP. A SINGLE
C PRECISION VERSION IS IDENTICAL WITH THIS PROGRAM EXCEPT FOR THE FACT THAT
C "IMPLICIT REAL*8 (A-H,O-Z)" IS NOT USED AND SINGLE PRECISION VERSIONS OF
C THE NSWC LIBRARY OF MATH ROUNTINES ARE USED INSTEAD OF THE DOUBLE PRECISION
C VERSIONS (REF. 8).
C USE LINK ESTIMATION2DP,DNRVG,FILE1 FOR LINKING ON THE VAX COMPUTER.
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON X(1000),BH(1000,1),U(1000)
      DIMENSION A(1000,1000),B(500500)
C FOR X(1000), USE B(500500), FOR X(2000), USE B(2001000)
      PI=3.1415926535897932
C SEE NRVG PROGRAM IN NSWC LIBRARY OF MATH ROUTINES (REF. 8) AND ALIGN THE NEXT
C FOUR VARIABLES TO CHECK THEIR MEANINGS. NOTE THAT THIS PROGRAM USES DRVG,
C THE DOUBLE PRECISION VERSION OF NRVG.
      M0=0
      NVEC=1
      KX=2000
      ISEED=2795601
C H IS THE FRACTIONAL BROWNIAN MOTION (FBM) PARAMETER, THETA0 IS THE INPUT
C PARAMETER OF THE MODEL DX(T)=(THETA0)X(T)DT+DBH(T) WHERE BH IS FBM
C PARAMETERIZED BY H.  THIS PROGRAM SIMULATES X(T) AFTER SIMULATING BH
C VIA THE NRVG ROUTINE.
      WRITE(6,*)'ENTER H,N=# OF INCREMENTS,DEL=INCREMENT,THETA0=THETA'
      WRITE(7,*)'ENTER H,N=# OF INCREMENTS,DEL=INCREMENT,THETA0=THETA'
      READ(5,*)H,N,DEL,THETA0
      WRITE(6,*)'H,N,DEL,THETA0',H,N,DEL,THETA0
      WRITE(7,*)'H,N,DEL,THETA0',H,N,DEL,THETA0
      M=N
C U IS THE MEAN VECTOR.
      DO I=1,M
        U(I)=0.
      ENDDO
      Z=2.-2.*H
C GAM=GAMMA FUNCTION (SEE REF. 8 FOR EXPLANATION OF THE DGAMMA FUNCTION GIVEN
C ON THE NEXT LINE).
      GAM=DGAMMA(Z)
      PIH=PI*H
      IF(H.EQ..5)THEN
        VH=1.0
      ELSE
C VH IS THE SCALE FACTOR IN THE COVARIANCE MATRIX OF BH.
        VH=-(GAM*COS(PIH))/(PIH*(2.*H-1.))
      ENDIF
      WRITE(6,*)'VH',VH
      WRITE(7,*)'VH',VH
C CALCULATE THE COVARIANCE MATRIX OF BH.
      DO I=1,N
        DO J=I,N
          TIME1=FLOAT(I)*DEL
          TIME2=FLOAT(J)*DEL
          S2H=ABS(TIME1)
          S2H=S2H**(2.*H)
          T2H=ABS(TIME2)
          T2H=T2H**(2.*H)
          TS=ABS(TIME2-TIME1)
          TS=TS**(2.*H)
          A(I,J)=(S2H+T2H-TS)*VH/2.
          A(J,I)=A(I,J)
        ENDDO
      ENDDO
C      WRITE(6,*) 'A'
C      WRITE(7,*) 'A'
C      DO I=1,N
C        WRITE(6,*) (A(I,J), J=1,N)
```

```
C         WRITE(7,*) (A(I,J), J=1,N)
C       ENDDO
C DMCVFS PACKS THE COVARIANCE MATRIX (REF 8).  THIS IS REQUIRED BY DNRVG.
        CALL DMCVFS(A,M,M,B)
C       WRITE(6,*) 'B',B
C       WRITE(7,*) 'B',B
C DNRVG SIMULATES FBM KNOWN AS BH IN THIS PROGRAM (REF 8).
      CALL DNRVG(M0,ISEED,NVEC,M,B,U,BH,KX,IERR)
      WRITE(6,*) 'M0,ISEED,NVEC',M0,ISEED,NVEC
      WRITE(6,*) 'M,KX',M,KX
C       WRITE(6,*)'U',U
C       WRITE(6,*) 'B',B
C       WRITE(6,*) 'BH VALUES', BH
      WRITE(6,*) 'IERR = ',IERR
      WRITE(6,*) 'U(1),U(100)',U(1),U(100)
      WRITE(7,*) 'M0,ISEED,NVEC',M0,ISEED,NVEC
      WRITE(7,*) 'M,KX',M,KX
C       WRITE(7,*)'U',U
C       WRITE(7,*) 'B',B
C       WRITE(7,*) 'BH VALUES', BH
      WRITE(7,*) 'IERR = ',IERR
      AV=0.
      DO I=1,N
        AV=AV+BH(I,1)
      ENDDO
C SAMPLE MEAN OF BH
      AV=AV/N
      WRITE(6,*)'SAMPLE MEAN OF BH',AV
      WRITE(7,*)'SAMPLE MEAN OF BH',AV
C STATE SIMULATES X(T) OF THE PARAMETRIC STOCHASTIC DIFFERENTIAL EQUATION.
C ESTIM IS THE PARAMETRIC ESTIMATOR OF THE STOCHASTIC DIFF. EQTN.
      CALL STATE(N,THETA0,DEL)
      CALL ESTIM(N,THETA,DEL)
C       WRITE(6,*) 'STATE MODEL VALUES', X
      WRITE (6,*) 'ACTUAL THETA', THETA0
      WRITE(6,*) 'ESTIMATED THETA', THETA
      WRITE(7,*) 'STATE MODEL VALUES', X
      WRITE (7,*) 'ACTUAL THETA', THETA0
      WRITE(7,*) 'ESTIMATED THETA', THETA
      STOP
      END
      SUBROUTINE STATE(N,THETA0,DEL)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL LNSUM,LNEXP,LNX
      COMMON X(2000),BH(2000,1),U(2000)
      SUM=0.
      AV1=0.
      ADD=0.
      ADDSQ=0.
C GO TO 10 IF THE ACTUAL PARAMETER IS NEGATIVE.
      IF(THETA0.LT.0)GO TO 10
      DO I=1,N
        REALI=FLOAT(I-1)
        REALJ=FLOAT(I)
        IF(I.EQ.1) THEN
          DBH=BH(1,1)
C         WRITE(7,*)'DBH',DBH
          POWER=0
        ELSE
          DBH=BH(I,1)-BH(I-1,1)
C         WRITE(7,*)'DBH',DBH
          POWER=-THETA0*REALI*DEL
        ENDIF
C AV1 AND ADD USED TO CALCULATE SAMPLE MEAN/VAR OF DBH FOR DEBUGGING THIS PROG.
        ADD=ADD+DBH
```

```
C ADDSQ AND ADD USED TO CALC SAMPLE VAR. FOR DEBUGGING.
        DBHSQ=DBH**2
        ADDSQ=ADDSQ+DBHSQ
        FUNC=EXP(POWER)
        TERM=FUNC*DBH
        SUM=SUM+TERM
C        WRITE(7,*)'SUM FOR LOG',SUM
C TAKE LOG IN ORDER TO HELP IN CASE OF LARGE SUM.
C THE FOLLOWING TAKES CARE OF FACT THAT ARG. OF LOG CANNOT BE NEG.
        IF(SUM.LT.0) THEN
          LNSUM=LOG(-SUM)
          ELSE
          LNSUM=LOG(SUM)
        ENDIF
        LNEXP=THETA0*REALJ*DEL
        LNX=LNEXP+LNSUM
        X(I)=EXP(LNX)
C NEXT LINE TAKES CARE OF CASE WHERE LOG OF -SUM WAS TAKEN.
       IF(SUM.LT.0) X(I)=-X(I)
C ABOVE GIVES X(T) AT T=I*DEL FOR THE CASE WHERE THETA0 IS NONNEGATIVE.
      ENDDO
C      WRITE(7,*) 'X VALUE',X
      AV1=ADD/N
      WRITE(6,*)'SAMPLE MEAN OF DBH',AV1
      WRITE(7,*)'SAMPLE MEAN OF DBH',AV1
      SQADD=ADD**2
      VARHAT=(N*ADDSQ-SQADD)/(N*(N-1))
      WRITE(6,*)'SAMPLE VAR OF DBH',VARHAT
      WRITE(7,*)'SAMPLE VAR OF DBH', VARHAT
      RETURN
 10   DO I=1,N
        SUM=0.
        DO J=1,I
        REALI=FLOAT(I)
        REALJ=FLOAT(J-1)
        IF(J.EQ.1) THEN
          DBH=BH(1,1)
C          WRITE(7,*)'DBH',DBH
          POWER=THETA0*REALI*DEL
        ELSE
          DBH=BH(J,1)-BH(J-1,1)
C         WRITE(7,*)'DBH',DBH
          POWER=THETA0*(REALI*DEL-REALJ*DEL)
        ENDIF
C AV1 AND ADD USED TO CALCULATE SAMPLE MEAN/VAR OF DBH FOR DEBUGGING THIS PROG.
        ADD=ADD+DBH
C ADDSQ AND ADD USED TO CALC SAMPLE VAR. FOR DEBUGGING.
        DBHSQ=DBH**2
        ADDSQ=ADDSQ+DBHSQ
        FUNC=EXP(POWER)
        TERM=FUNC*DBH
        SUM=SUM+TERM
        ENDDO
      X(I)=SUM
C THIS GIVES X AT T=I*DEL FOR THE CASE WHERE THETA0 IS NEGATIVE.
      ENDDO
      NN=N*(N+1)/2
      AV1=ADD/NN
      WRITE(6,*)'SAMPLE MEAN OF DBH',AV1
      WRITE(7,*)'SAMPLE MEAN OF DBH',AV1
      SQADD=ADD**2
C     VARHAT=(NN*ADDSQ-SQADD)/(NN*(NN-1))
C     WRITE(6,*)'SAMPLE VAR OF DBH',VARHAT
C     WRITE(7,*)'SAMPLE VAR OF DBH', VARHAT
      RETURN
```

```
      END
      SUBROUTINE ESTIM(N,THETA,DEL)
C THIS IS A GENERALIZED VERSION OF THE LEAST SQUARES ESTIMATOR FOR THE PARAMETER
C THETA IN A STOCHASTIC DIFFERENTIAL EQUATION DX(T)=(THETA)X(T)DT+DBH(T). SEE
C THE TEXT OF THIS REPORT FOR AN EXPLANATION.
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON X(2000),BH(2000,1),U(2000)
      SUM1=0.
      SUM2=0.
      DO I=1,N
        IF(I.EQ.1)THEN
          XTERM1=0.
          DX=X(I)
        ELSE
          DX=X(I)-X(I-1)
          XTERM1=X(I-1)*DX
        ENDIF
        SUM1=SUM1+XTERM1
C SUM1 IS THE INTEGRAL OF THE PROCESS X WITH RESPECT TO THE PROCESS
C X ITSELF.  INTEGRATION IS FROM 0 TO TIME N*DEL.
      ENDDO
      DO J=1,N
        IF(J.EQ.1)THEN
          XTERM2=0.
        ELSE
          XSQ=X(J-1)**2.
          XTERM2=XSQ*DEL
        ENDIF
C SUM2 IS THE INTEGRAL OF THE SQUARE OF THE PROCESS X WITH RESPECT TO TIME.
C INTEGRATION IS FROM 0 TO TIME N*DEL.
        SUM2=SUM2+XTERM2
      ENDDO
      WRITE(6,*)'SUM1',SUM1
      WRITE(6,*)'SUM2',SUM2
      WRITE(7,*)'SUM1',SUM1
      WRITE(7,*)'SUM2',SUM2
      THETA=SUM1/SUM2
C NOTE THAT IN BOTH INTEGRALS RESPRESENTED BY SUM1 AND SUM2 RESPECTIVELY
C ARE APPROXIMATED WHERE IN EACH TERM OF THE SUM, THE INTEGRAND IS APPROX.
C AS THE I-1 TERM. THIS IS TO MIMIC THE ITO INTEGRAL.  SINCE THESE ARE
C NOT TRUE ITO INTEGRALS, IT IS NOT NECESSARY TO CHOOSE THE TERM AT I-1.
      RETURN
      END
```

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